FINAL: ALGEBRA III

Date: **21st December 2020**

Please do not consult anyone. You may use the text books or class notes. Please do not use any other resource.

The Total points is **85**. The maximum you can score is 80.

(1) (20 points) Let $R = (\mathbb{Z}/12\mathbb{Z})[x]$ be the polynomial ring over $\mathbb{Z}/12\mathbb{Z}$. Compute the nilradical of R. Prove or disprove the following statement. Every nonzero R-module contains a nonzero torsion element.

(2) (20 points) Let $R = \mathbb{Q}[x, y, z]$ be a polynomial ring in 3 variables, $I = (x, y)$, $J = (z)$ and $m = (x, y, z)$ be ideals of R. Let $M = m/J \oplus R[1/x]$. Show that M/IM is a finite dimensional $\mathbb{Q}$-vector space. Also compute its dimension.

(3) (20 points) Determine whether the following rings are an integral domain, a PID or a UFD. Justify your answer.
  (a) $A = (\mathbb{Z}/45\mathbb{Z})[x, y, z]/(3x - 1, z^3x - x^2yz + x^3y)$
  (b) $B = (\mathbb{Z}/5\mathbb{Z})[x, y, z]/(3x - 1, z^3x - x^2yz + x^3y)$
  (c) $C = (\mathbb{Z}/9\mathbb{Z})[x, y, z]/(3x - 1, z^3x - x^2yz + x^3y)$
  (d) $D = \mathbb{Z}[x, y, z]/(3x - 1, z^3x - x^2yz + x^3y)$

(4) (25 points) Let $V_1 = \mathbb{C}[y]/(y^2(y^2 - 1)(y^3 - 1))$, $V_2 = \mathbb{C}[z]/(z^3 - z^2)$ and $V = V_1 \oplus V_2$ be $\mathbb{C}$-vector spaces. Let $\phi_1 : V_1 \to V_1$, $\phi_2 : V_2 \to V_2$ and $\phi : V \to V$ be given by $\phi_1(v_1) = \bar{y}v_1$, $\phi_2(v_2) = \bar{z}v_2$ and $\phi(v_1, v_2) = (\bar{y}v_1, \bar{z}v_2)$ $\forall v_1 \in V_1, \forall v_2 \in V_2$ respectively. Find the rational canonical and the Jordan canonical forms of ϕ_1, ϕ_2 and ϕ.